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A game-theoretic implication of the Riemann hypothesis☆

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ABSTRACT

The Riemann hypothesis (RH) is one of the major unsolved problems in pure mathematics. In the present paper, a parameterized family of non-cooperative games is constructed with the property that, if RH is true, then any game in the family admits a unique Nash equilibrium. We argue that this result is not degenerate. Indeed, neither is the conclusion a tautology, nor is RH used to define the family of games.

1. Introduction

The Riemann hypothesis (RH) is one of the major unsolved problems in pure mathematics and, more specifically, in the field of analytic number theory. Put forth as a conjecture by the famous Göttingen-based mathematician Bernhard Riemann (Riemann, 1859), the hypothesis concerns the *zeta function*,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}\, s > 1), \tag{1}$$

where n runs over all positive integers, and the real part of s is required to exceed one to guarantee convergence of the infinite sum. The zeta function is known to admit an analytic[1] extension to the complex plane, with a simple pole at $s = 1$ as an exception, and trivial zeros at all negative even integers $s = -2, -4, \dots$ (Titchmarsh, 1986). RH says that the other, non-trivial zeros all lie on the *critical line* defined through $\mathrm{Re}\, s = \frac{1}{2}$. If true, the conjecture would admit powerful conclusions about the distribution of prime numbers (Davenport, 1980). Despite the intuitive nature of RH and an extensive body of numerical support for it, a formal proof has remained elusive up to the present day.[2]

The purpose of the present paper is to report on a somewhat curious observation that relates RH to the theory of non-cooperative games. More specifically, a class of two-player games with continuous strategy sets will be constructed that, provided that RH is true, all feature a unique Nash equilibrium in randomized strategies. We will also argue that our result is not degenerate in any obvious way.

The games considered below have some similarity to standard models of competition known as contests (Konrad, 2009). In fact, our games roughly fall into the class of so-called *difference-form contests* (Hirshleifer, 1989; Baik, 1998). In general, in a difference-form contest, two players independently and simultaneously choose a nonnegative level of costly effort. Then, the difference between the respective efforts chosen by players 1 and 2 is compared to a random term that is symmetrically distributed about zero (and not known to the players at the time of decision making). If the difference turns out to strictly exceed the random term, then the prize goes to player 1. If, however, the difference is less than the random term, then the prize goes to player 2. Finally, in the case of a tie, i.e., if the difference is equal to the random term, then each player has an equal chance of receiving the prize. The equilibrium analysis of such contest models has been found useful to study the nature of military conflict, for instance.

As recently noted by the author (Ewerhart, 2021), the Nash equilibrium in a difference-form contest is unique if the density function f that governs the distribution of randomness is both analytic and a

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[1] A function f is *analytic* if the derivatives of f of all finite orders exist and, at any point, f is locally approximated arbitrarily well by its Taylor expansion.

[2] Proving RH is one of the seven problems the solution of which the Clay Mathematics Institute would award a prize of one million dollars (Bombieri, 2000). Numerous interesting but ultimately partial results are available. For example, it is known that "more than 40 percent" of the non-trivial zeros of ζ lie on the critical line (Conrey, 1989). Moreover, starting with Turing (1953), substantial effort has been invested into attempts to reject RH using computational means. However, at least the first 10^{13} non-trivial zeros lie exactly on the critical line (Gourdon, 2004; Platt and Trudgian, 2021; see also Edwards, 1974, Ch. 8). Still, according to Sarnak (2004, pp. 6–7), this need not mean that RH is "likely true." At the time of writing, however, RH remains an open mathematical problem.

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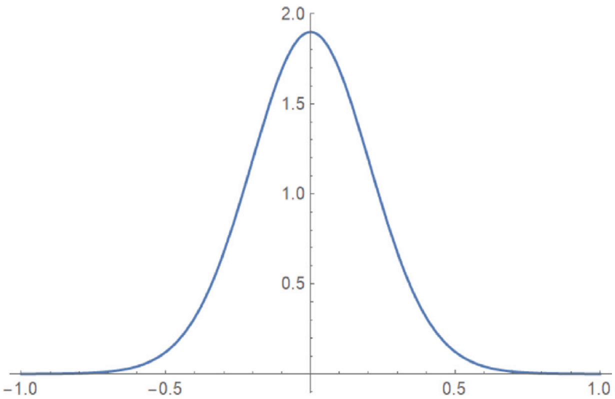

Fig. 1. The graph of $f^{\#}$.

proper Pólya frequency function.[3] Examples of functions that simultaneously satisfy both assumptions include any logistic or normal probability densities (Karlin, 1957, 1959). Therefore, Nash equilibria in difference-form contests with logistic or normal noise are unique. In contrast, the uniform density is neither analytic nor a proper Pólya frequency function. And indeed, as will be shown by example later in the paper, multiple equilibria are possible in the model with uniform noise.

Given the salience of the property of being a proper Pólya frequency function for equilibrium uniqueness in difference-form contests, it is natural to explore the set of probability densities that possess this property. In an interesting paper, Gröchenig (2020) introduced a special function that is a Pólya frequency function if and only if RH is true. Up to a constant factor, that function is given by

$$f^{\#}(t) = \frac{\xi(\frac{1}{2})}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)dx}{\xi(\frac{1}{2}+x)} \qquad (-\infty < t < \infty), \tag{3}$$

where

$$\xi(s) = \tfrac{s(s-1)}{2}\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) \tag{4}$$

is the ξ-function, a close relative of the zeta function (1), and $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx$ is the gamma function. The graph of $f^{\#}$ is outlined in Fig. 1.[4]

Our class of games consists of generalized difference-form contests in which $f^{\#}$, interpreted as a probability density, governs the underlying distribution of noise. We first check that $f^{\#}$ is measurable and symmetric with respect to the origin, which will be assumptions in our definition of a generalized difference-form contest. Then, we show that $f^{\#}$ is analytic. Finally, strengthening one of the key results in Gröchenig (2020), we prove that $f^{\#}$ is actually a *proper* Pólya frequency function if RH is true. From these observations, the main result of the present paper is derived using, in particular, the sufficient condition for equilibrium uniqueness mentioned above.

To explain why we believe that our example is not degenerate, we add a discussion, which consists of two parts. First, it is shown with the help of an example that equilibrium uniqueness is not generally satisfied in the class of difference-form contests. Thus, the conclusion of our result need not be true in the absence of the premise. In the second part of the discussion, we present examples of games that are defined using RH and explain how our examples differ in nature from such degenerate cases.

Literature notes. There does not seem to exist prior academic work that connects RH to the theory of games. Nobel laureate John Nash, whose contributions in the early 1950s became the basis of modern game theory (Nash, 1950, 1951) and who had also solved Hilbert's 19th problem on partial differential equations, is understood to have worked on RH.[5] However, Milnor's (1998) bibliography of Nash's work does not list any manuscript written by Nash with an obvious relationship to number theory.[6] Games with analytic payoff functions appear in early work on two-person zero-sum games on the square (Karlin, 1957, 1959). See also Ewerhart (2015) and Levine and Mattozzi (2022). Two-player difference-form contests have been studied by Hirshleifer (1989), Baik (1998), Che and Gale (2000), Ewerhart and Sun (2018), and Ewerhart (2021), in particular.[7]

The remainder of this paper is structured as follows. Section 2 introduces a class of generalized difference-form contests. Section 3 presents the main result. Section 4 offers some discussion. Section 5 concludes. Technical proofs have been relegated to an Appendix.

2. Difference-form contests

In this section, we introduce a class of generalized difference-form contests (Section 2.1), offer some examples (Section 2.2), and provide conditions sufficient for the existence and uniqueness of a symmetric Nash equilibrium (Section 2.3).

2.1. Set-up and notation

Two players, referred to as player 1 and player 2, each choose a nonnegative effort, $x_1 \geq 0$ and $x_2 \geq 0$. There is a prize of value $W > 0$. Payoffs are given by

$$\Pi_1(x_1, x_2) = F(x_1 - x_2)W - x_1,$$
$$\Pi_2(x_1, x_2) = F(x_2 - x_1)W - x_2,$$

where $F(t) = \frac{1}{2} + \int_0^t f(\tau)d\tau$ for some measurable function f that is symmetric with respect to the origin.[8] The resulting game will be denoted by $G_0 \equiv G_0(f, W)$.

[3] Following Schoenberg (1947, 1951) and Schoenberg and Whitney (1953), an integrable function $f \geq 0$ with $\int_{-\infty}^{\infty} f(x)dx > 0$ is called a *Pólya frequency function* if, for any $n \geq 1$, and for any real parameters $a_1 > a_2 > \cdots > a_n$ and $b_1 > b_2 > \cdots > b_n$, we have

$$\begin{vmatrix} f(a_1-b_1) & f(a_1-b_2) & \cdots & f(a_1-b_n) \\ f(a_2-b_1) & f(a_2-b_2) & \cdots & f(a_2-b_n) \\ \vdots & \vdots & \ddots & \vdots \\ f(a_n-b_1) & f(a_n-b_2) & \cdots & f(a_n-b_n) \end{vmatrix} \geq 0. \tag{2}$$

A function f is a *proper Pólya frequency function* if it satisfies the definition above with strict inequality in (2). The strict conditions on the determinant of the translation matrix for $n = 1$ and $n = 2$ can be shown to correspond to the positivity and strict logconcavity of f, respectively, but the conditions for $n \geq 3$ do not admit an equally simple interpretation.

[4] All computations have been conducted using Wolfram's *Mathematica* 12.0.0.

[5] According to a popular biography (Nasar, 1998), as well as to a Hollywood movie based upon it, Nash's presentation on the topic at Columbia University in 1959 became incomprehensible because of his beginning mental illness (see also Sabbagh, 2003).

[6] In a volume coedited by John Nash, Connes (2016) related RH to chip-firing games on graphs. Those games, however, are one-player "solitaire" problems (Baker and Norine, 2007). More recently, Carmona et al. (2020) recovered the Gaussian unitary ensemble (GUE) distribution from limits of equilibria in N-player stochastic games as $N \to \infty$. While GUE describes the distribution of distances between neighboring zeros of the Riemann zeta function (Montgomery, 1973; Odlyzko, 1987; Rudnick and Sarnak, 1994, 1996), GUE arises in numerous other applications as well.

[7] For an introduction to total positivity, see Karlin (1968). Katkova (2007) related RH to totally positive sequences.

[8] To see that payoffs are well-defined, one notes that F is defined also for negative arguments because

$$F(-t) = \frac{1}{2} + \int_0^{-t} f(\tau)d\tau = \frac{1}{2} - \int_{-t}^0 f(\tau)d\tau = \frac{1}{2} - \int_0^t f(\tau)d\tau.$$

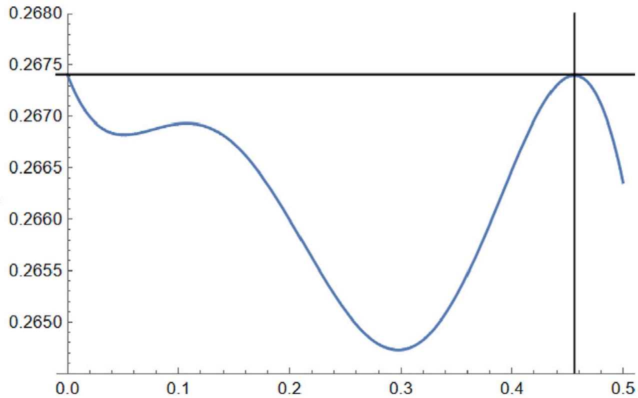

Fig. 2. Expected payoff against the equilibrium strategy.

In contrast to standard developments (Hirshleifer, 1989; Baik, 1998), the above definition does not require f to be a probability density function. If f is a probability density function, however, then the assumption of symmetry implies that $F(t) = \int_{-\infty}^{t} f(\tau)d\tau$ is indeed the corresponding distribution function. Moreover, $F(x_1 - x_2)$ is the probability that player 1 wins the prize, while $F(x_2-x_1) = 1-F(x_1-x_2)$ is the probability that player 2 wins the prize. In that case, therefore, G_0 admits an interpretation as a symmetric *difference-form contest*. If f is not a probability density, however, then G_0 is not a difference-form contest in the usual meaning of the term.

Further, it should be noted that the above payoff functions need, in general, not be quasiconcave in own effort even if f is a well-behaved probability density (e.g., symmetric, continuously differentiable, and strictly unimodal). The reason is that F will then be strictly convex for negative arguments, which may lead to local minima in a player's payoff function if the opponent chooses a positive effort. It is therefore natural to allow for mixed strategies, defined as probability distributions on the Borel subsets of a suitably chosen compact subinterval of the real line (Dasgupta and Maskin, 1986). If f is a density function, then effort levels weakly exceeding W are strictly dominated by zero effort. We therefore choose this interval to be $[0, W]$ for both players.

2.2. Examples

The following two examples illustrate equilibria in difference-form contests.

Example 1. (Hirshleifer, 1989; Baik, 1998) Let f be a continuous density single-peaked at the origin. If $W \leq 1/f(0)$, then players' payoffs are strictly declining in their respective own strategy. In that case, therefore, the unique Nash equilibrium of $G_0 = G_0(f, W)$ is in pure strategies (i.e., in degenerate probability distributions), with equilibrium efforts given by $x_1^* = x_2^* = 0$.

Example 2. (Ewerhart and Sun, 2018) Let f be the logistic density $f(t) = \frac{\alpha \exp(-\alpha t)}{(1+\exp(-\alpha t))^2}$, where $\alpha = 6.75$. Suppose also that $W = 1$. Then, the unique mixed-strategy equilibrium in $G_0 = G_0(f, W)$ is symmetric, and has each player $i \in \{1, 2\}$ independently choose $x_i = y_1 \equiv 0.45597$ with probability $q_1 = 0.51011$, and $x_i = 0$ with probability $q_2 = 1 - q_1$. The equilibrium payoff is $\Pi_i^* = 0.2674$. Player i's expected payoff against the equilibrium strategy,

$$E[\Pi_i(x_1, x_2)] = \frac{q_1}{1 + \exp(-\alpha(x_i - y_1))} + \frac{q_2}{1 + \exp(-\alpha x_i)} - x_i,$$

considered as a function of x_i, is depicted in Fig. 2. As can be seen, its global maxima are located at $x_i = y_1$ and at $x_i = 0$, respectively.

In general, however, the equilibrium of a difference-form contest need not be unique. An example will be provided later in the paper (see Section 4).

2.3. Conditions for existence and uniqueness of an equilibrium

We will make use of the following result.

Lemma 1. *Suppose that f is a probability density function on $\mathbb{R}$ that is symmetric with respect to the origin, and let $W > 0$. Then, a symmetric equilibrium μ^* exists in $G_0 = G_0(f, W)$. If, in addition, f is both analytic and a proper Pólya frequency function, then there are no asymmetric equilibria, and μ^* is the unique equilibrium in G_0.*

Proof. See the Appendix.

The existence part of the proof of Lemma 1 is based on standard conditions for symmetric compact games with continuous payoffs (Becker and Damianov, 2006). In contrast, the uniqueness part, adapted from Ewerhart (2021), requires several steps in which our assumptions are combined with results from both complex analysis and the theory of two-person zero-sum games. We will also see below that the assumptions of Lemma 1 cannot be easily dropped.

How does the assumption that f is a proper Pólya frequency function contribute to the conclusion that the equilibrium is unique? General properties of contests with analytic payoffs imply the existence of a finite set S of pure strategies with the property that *any* mixed equilibrium strategy randomizes over this set.[9] Then, given the finite set of potential maximizers, necessary first-order conditions hold at all positive effort levels used with positive probability in equilibrium. Moreover, if a player chooses a zero effort level with positive probability (this is always the case for both players, as a consideration of second-order conditions shows), then the resulting equilibrium payoff Π^* must necessarily be the same as the expected payoff resulting from the lowest positive effort level potentially used with positive probability (and if there is no such positive effort level, then there is nothing to show). These first-order conditions and the indifference condition are combined into a system of linear equations, one for each player, in which the probabilities with which pure strategies are used and the player's equilibrium payoff are the unknowns. To prove uniqueness, it then suffices to show that this system of linear equations is not degenerate.

To this end, one exploits the assumption that f is a proper Pólya frequency function. In fact, given that f features prominently in the first-order conditions, this last point would be straightforward if it were known that all effort levels used in equilibrium with positive probability are positive. In that case, the unique solvability of the linear system of first-order conditions would follow directly from the fact that the determinant of the relevant translation matrix is positive. However, as mentioned above, both players necessarily choose a zero effort with positive probability, which is why this simple argument does not go through. Instead, the linear system of equations contains an indifference condition, which involves the distribution function F in addition to the density function f. But then, fortunately, the sign of the relevant determinant may still be evaluated as an integral over positive determinants.

3. The main result

This section is central to the analysis. We start by stating the main result (Section 3.1). Then, we provide an overview of the proof (Section 3.2).

[9] In a nutshell, the discreteness of the support of any best response is a consequence of the identity theorem for analytic functions and the fact that analytic payoff functions remain analytic under the formation of expectations (Ewerhart, 2015). The existence of the finite set S follows then from the exchangeability of equilibrium strategies in contests, which are strategically zero-sum (Ewerhart, 2017).

3.1. Statement of the main result

In his study of the properties of the zeta function, Riemann (1859) showed that the ξ-function (4) is analytic on the entire complex plane and satisfies the astounding functional equation

$$\xi(s) = \xi(1 - s).$$

From the functional equation, it is immediate that the "shifted" ξ-function $x \mapsto \xi(\frac{1}{2} + x)$ is symmetric with respect to a reflection at the origin. Moreover, from Stirling's asymptotic formula for the gamma function, it is known that $\xi(s)$ does not possess any zero on the real line and grows exponentially as $s \to \pm\infty$. As pointed out by Gröchenig (2020, Thm. 4), these properties of the ξ-function imply that the Fourier transform of $1/\xi(\frac{1}{2} + x)$,

$$\widetilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(itx)}{\xi(\frac{1}{2} + x)} dx$$

is well-defined. Moreover, as mentioned above, he also showed that $\widetilde{f}$ is a Pólya frequency function if and only if RH is true.

In the Appendix, we show that $f^{\#}(t) = \xi(\frac{1}{2})\widetilde{f}(t)$, where $t \in \mathbb{R}$. Clearly, $f^{\#}$ does not belong to the class of functions commonly employed in economics and statistical analysis; for example, it does not appear in Johnson et al. (1995). In fact, little definite is known about $f^{\#}$. For instance, it is not even known if $f^{\#}$ is globally nonnegative or if its integral over the real line is finite. We establish the following properties of $f^{\#}$.

Lemma 2. *The function $f^{\#} = \xi(\frac{1}{2})\widetilde{f}$ is real-valued on $\mathbb{R}$. Moreover,*
(i) $f^{\#}$ is measurable and symmetric with respect to the origin;
(ii) $f^{\#}$ is analytic.

Proof. See the Appendix. □

By collecting elementary properties of $f^{\#}$, part (i) ensures that $f^{\#}$ defines a generalized difference-form contest, as introduced in Section 2. The generalized difference-form contest in which f is chosen to be $f^{\#}$ will be denoted by $G_0^{\#} \equiv G_0(f^{\#}, W)$.

Part (ii) says that $f^{\#}$ is real-analytic. This property is needed to apply Lemma 1 in the proof of our main result below. The proof of analyticity exploits the before-mentioned property that the reciprocal of the "shifted" ξ-function diminishes exponentially for real arguments large in absolute value.[10]

We are ready to state our main result.

Theorem 1. *If RH is true, then $G_0^{\#} = G_0(f^{\#}, W)$ admits precisely one equilibrium for any $W > 0$.*

Proof. See the next subsection. □

3.2. Proof of Theorem 1

Given Lemmas 1 and 2, the main point to prove is that $f^{\#}$ is a proper Pólya frequency function if RH is true. But we also need to ensure that $f^{\#}$ is a probability density in that case. These facts are established by the following lemma.

Lemma 3. *If RH is true, then $f^{\#}$ is both a proper Pólya frequency function and a probability density function.*

Proof. See the Appendix. □

Lemma 3 is based upon a mathematically deep correspondence between Pólya frequency functions on the one hand and functions of the Laguerre-Pólya class of type II (i.e., functions analytic on the complex plane that are locally uniform limits, not identically zero, of polynomials whose zeros all lie on the real line) on the other. This correspondence was established by Schoenberg (1947) who observed that Pólya frequency functions may be characterized as normally smoothed limits of convolutions of multiple exponential probability density functions. Since the (two-sided) Laplace transform of an exponential probability density function is just the reciprocal of a factor in a Weierstrass product representation (Schoenberg, 1951, pp. 349–350), and since the Laplace transform converts convolutions into products, that correspondence connects Pólya frequency functions with functions of the Laguerre-Pólya class. In particular, as noted by Gröchenig (2020, Thm. 4), the shape of $\widetilde{f}$ allows conclusions regarding the location of the zeros of the ξ-function, and vice versa. More specifically, $\widetilde{f}$ is a Pólya frequency function if and only if RH is true. Lemma 3 strengthens the "if" direction of that equivalence using a result by Schoenberg and Whitney (1953) for proper Pólya frequency functions.

4. Discussion

In this section, we will argue that Theorem 1 is not degenerate, because the conclusion is not a tautology (Section 4.1) and because RH does not enter the definition of the game (Section 4.2).

4.1. The conclusion of Theorem 1 is not a tautology

Clearly, it is not at all difficult to come up with a non-cooperative game that has precisely one Nash equilibrium if RH is true. For example, the Prisoner's Dilemma admits a unique Nash equilibrium if RH is true.[11] Indeed, given that the conclusion is true, the implication holds regardless of the validity of the premise (by the rules of Boolean logic). But the situation is different here, i.e., the conclusion of Theorem 1 is not a tautology. To see why, we illustrate the possibility of multiple equilibria by modifying the assumptions in the framework of Che and Gale (2000).

Example 3. Let f be the uniform density over the interval $[-c, c]$, for some $c > 0$. Thus, $f(t) = \frac{1}{2c}$ if $t \in [-c, c]$, and $f(t) = 0$ otherwise. Then, $x_1^* = c$ and $x_2^* = 0$ form an asymmetric Nash equilibrium in pure strategies in $G_0 = G_0(f, W)$ in the *non-generic* case $W = 2c$. Indeed, with

$$F(t) = \begin{cases} 0 & \text{if } t < -c \\ \frac{1}{2} + \frac{t}{2c} & \text{if } t \in [-c, c] \\ 1 & \text{if } t > c, \end{cases}$$

the payoff functions against the respective opponent's equilibrium strategies are given by

$$\begin{aligned} \Pi_1(x_1, 0) &= 2cF(x_1) - x_1 \\ &= \begin{cases} c & \text{if } x_1 \in [0, c] \\ 2c - x_1 & \text{if } x_1 > c, \end{cases} \\ \Pi_2(c, x_2) &= 2cF(x_2 - c) - x_2 \\ &= \begin{cases} 0 & \text{if } x_2 \in [0, 2c] \\ 2c - x_2 & \text{if } x_2 > 2c. \end{cases} \end{aligned}$$

An inspection of these relationships shows that $\Pi_1(c, 0) \geq \Pi_1(x_1, 0)$ for any $x_1 \geq 0$, and $\Pi_2(c, 0) \geq \Pi_2(c, x_2)$ for any $x_2 \geq 0$. Hence, $(x_1^*, x_2^*) = (c, 0)$ is indeed an equilibrium. By symmetry, a second equilibrium is given by $(x_1^{**}, x_2^{**}) = (0, c)$.

[10] By extending the proof of Lemma 2, one can show that $f^{\#}$ is also uniformly continuous, positive definite, and tending to zero for large arguments. However, these facts are not needed below.

[11] I am grateful to John Levy for providing this example.

G_1	L	R
T	0,0	0,0
B	0,0	-1,-1

G_2	L	R
T	1,θ	θ,0
B	0,1	0,0

Fig. 3. The games G_1 and G_2.

In view of the possibility of multiple equilibria illustrated by Example 3, it is not known (and might never be known) if the conclusion of Theorem 1 is true or false. Thus, the conclusion is an open conjecture like the hypothesis.[12] In particular, if the conclusion of equilibrium uniqueness could be shown to be wrong (which is not feasible in the case of the Prisoner's Dilemma), then the hypothesis would be proven wrong.[13]

4.2. RH does not enter the definition of the game

It would be nice, and certainly more satisfying, to find a game-theoretic conjecture that is logically equivalent to RH. In the abstract, this is actually not a big problem. E.g., one may even easily write down games for which the existence of a unique Nash equilibrium is equivalent to RH.

To see this, consider the game G_1 depicted in Fig. 3, where

$$\theta = \begin{cases} +1 & \text{if RH is true} \\ -1 & \text{if RH is false.} \end{cases}$$

If RH is true, then G_1 admits (T,L) as the unique Nash equilibrium. If, however, RH is false, then there are two Nash equilibria in pure strategies, viz. (T,R) and (B,L). Similarly, G_2 admits (T,L) as a unique pure-strategy Nash equilibrium if RH is true, and otherwise no pure-strategy Nash equilibrium.

In such examples, however, RH is used directly in the description of the payoff functions. That is, even if the strategy chosen by player 2 is correctly anticipated in G_1 or G_2, a human player 1 may not be able to tell if T yields a higher payoff than B. In contrast, RH has no role in the definition of payoffs in $G_0^{\#}$, i.e., expected payoffs could, at least in principle, be approximated up to arbitrary accuracy without assuming RH. In fact, given that best responses in the mixed extension of $G_0^{\#}$ have finite support, this argument extends to the relevant class of randomized strategies. For this reason, the two-player contest might be a more appealing example than G_1 or G_2, even though Theorem 1 does not capture a logical equivalence.

5. Concluding remarks

We presented an example of a parameterized family of two-person games with the property that every game in this family admits a unique Nash equilibrium if RH is true. Thus, in principle, a game theorist capable of identifying two distinct equilibria in just one of the considered games would reject RH. We also explained why we believe that our result is not degenerate in any obvious way.

As discussed, it would be desirable to better understand how tight the conditions of Theorem 1 are for equilibrium uniqueness. For example, as suggested by an anonymous referee, it would be interesting to know if the existence of a non-trivial zero of the Riemann zeta function slightly off the critical line, possibly with a very large absolute imaginary component, would allow to construct multiple equilibria in some game $G_0^{\#} = G_0(f^{\#}, W)$ for some $W > 0$.[14] Such difficult questions, however, must be left for future work.

Data availability

No data was used for the research described in the article.

Appendix

This appendix contains proofs omitted from the body of the paper.

Proof of Lemma 1 (*Existence*)**.** Let $W > 0$. By assumption, each player's set of pure strategies is the interval $[0, W]$, hence compact and convex. Moreover, payoff functions are continuous. Therefore, making use of Becker and Damianov (2006, Thm. 1), $G_0 = G_0(f, W)$ admits a symmetric mixed-strategy Nash equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$, where $\mu_1^* = \mu_2^*$.

(Uniqueness) We check the conditions of Ewerhart (2021, Prop. 1).[15] By assumption, f is a proper Pólya frequency function. Hence, in particular, f is a Pólya frequency function. Using Schoenberg (1951, Lemma 2), this implies that f is logconcave. But given analyticity, f is differentiable. Moreover, $f > 0$ because f is a proper Pólya frequency function. Therefore, f'/f is weakly declining. Next, one notes that $f'(0) = 0$ because f is symmetric with respect to the origin. But, since f is analytic on $\mathbb{R}$, so is f'. Further, f' is not constant (otherwise f would be affine, in conflict with the assumption that f is a probability density function). Hence, $t = 0$ is an isolated zero of f'. Combining this with the fact that f'/f is weakly declining implies that $f'(t)/f(t) < 0$ for all $t > 0$, and $f'(t)/f(t) > 0$ for all $t < 0$. This means that all conditions of Ewerhart (2021, Prop. 1) are satisfied. Thus, the equilibrium is indeed unique. □

For the reader's convenience, the material below has been adapted from Ewerhart (2021). Compared to the original contribution, however, the proof below is shorter because the difference-form contest is assumed to be symmetric in the present analysis.

Self-contained proof of the uniqueness part of Lemma 1. Fix $W > 0$. As shown above, there is at least one (even symmetric) mixed-strategy Nash equilibrium $\mu^* = (\mu_1^*, \mu_2^*)$ in $G_0 = G_0(f, W)$. Given that F is analytic on the real line, a straightforward extension of Ewerhart and Sun (2018, Lemma 1) shows that there is a finite set $S \subseteq [0, W]$ of pure strategies such that any pure best response to μ_1^* is contained in S. Next, we note that $\mu_2^* = \mu_1^*$ is a mixed best response to μ_1^*. Hence, the support of μ_1^* is finite and contained in S. Suppose that there exists another equilibrium $\mu^{**} = (\mu_1^{**}, \mu_2^{**})$ in G_0. By subsidizing each player with the other player's effort cost, the game G_0 is seen to be strategically equivalent to a two-person zero-sum game, which implies

[12] Both RH and equilibrium uniqueness in the two-player contest may be characterized as being undecidable in the *practical* sense. Undecidability in the *logical* sense is a possibility here as well (i.e., RH and/or equilibrium uniqueness might be true but not provable), but this possibility is not crucial for the present discussion.

[13] A similar type of reasoning is used in the literature on the P versus NP problem in computational complexity theory (Cook, 1971), which happens to be a millennium problem like RH.

[14] On a more speculative note, one might want to extend the analysis to more general classes of L-functions (Sarnak, 2004). However, additional assumptions would be needed, such as that the L-function does not vanish at $s = \frac{1}{2}$ (Stark and Zagier, 1980).

[15] A self-contained proof is added below.

the exchangeability of Nash equilibrium strategies (Ewerhart, 2017, Lemma A.1).[16] Therefore, both μ_1^* and μ_1^{**} are mixed best responses to μ_1^* and, by symmetry of the game, also to μ_1^{**}. Consider the set of pure strategies $S' = \{y_1 > y_2 > \cdots > y_K \geq 0\}$ in the support of either μ_1^* or μ_1^{**} (or both).[17] If $K = 1$, then $\mu_1^* = \mu_1^{**}$ and we are done. Suppose, therefore, that $K \geq 2$. Equating marginal benefits with marginal costs at the certainly positive levels of effort used in equilibrium $y_1, \ldots, y_{K-1}$, we have the necessary first-order conditions,

$$\sum_{\hat{k}=1}^{K} q_{\hat{k}} f(y_k - y_{\hat{k}}) W = 1 \qquad (k = 1, \ldots, K-1), \tag{5}$$

where $q_1, \ldots, q_K$ denote the respective probabilities with which pure strategies $y_1, \ldots, y_k$ are used in equilibrium. Moreover, we know that

$$-y_k + \sum_{\hat{k}=1}^{K} q_{\hat{k}} F(y_k - y_{\hat{k}}) W = \Pi^* \qquad (k \in \{K-1, K\}), \tag{6}$$

where Π^* is the equilibrium payoff resulting from μ^* (analogous conditions for $k \in \{1, \ldots, K-2\}$ are not needed, neither is the accounting equation $\sum_{\hat{k}=1}^{K} q_{\hat{k}} = 1$ needed). Combining the $(K-1)$ first-order conditions (5) with the two payoff conditions (6), we arrive at the system

$$M \begin{pmatrix} q_1 \\ \vdots \\ q_K \\ -\Pi^*/W \end{pmatrix} = \frac{1}{W} \begin{pmatrix} 1 \\ \vdots \\ 1 \\ y_{K-1} \\ y_K \end{pmatrix} \in \mathbb{R}^{K+1},$$

with the square matrix $M \in \mathbb{R}^{(K+1)\times(K+1)}$ defined by

$$M = \begin{pmatrix} \underbrace{f(y_1 - y_1)}_{=f(0)} & \cdots & f(y_1 - y_{K-1}) & f(y_1 - y_K) & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ f(y_{K-1} - y_1) & \cdots & \underbrace{f(y_{K-1} - y_{K-1})}_{=f(0)} & f(y_{K-1} - y_K) & 0 \\ F(y_{K-1} - y_1) & \cdots & \underbrace{F(y_{K-1} - y_{K-1})}_{=1/2} & F(y_{K-1} - y_K) & 1 \\ F(y_K - y_1) & \cdots & F(y_K - y_{K-1}) & \underbrace{F(y_K - y_K)}_{=1/2} & 1 \end{pmatrix}.$$

We claim that this system admits at most one solution. Indeed, subtracting the last row from the second-to-last row leads to

$$\det M = \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ F(y_{K-1} - y_1) - F(y_K - y_1) & \cdots & F(y_{K-1} - y_K) - \frac{1}{2} \end{vmatrix}.$$

Next, developing the determinant along the last row yields

$$\det M = \sum_{k=1}^{K} \{(-1)^{k+K} (F(y_{K-1} - y_k) - F(y_K - y_k)) \times \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_{k-1}) & f(y_1 - y_{k+1}) & \cdots & f(y_1 - y_K) \\ \vdots & & \vdots & \vdots & & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_{k-1}) & f(y_{K-1} - y_{k+1}) & \cdots & f(y_{K-1} - y_K) \end{vmatrix} \}.$$

Using

$$F(y_{K-1} - y_k) - F(y_K - y_k) = \int_{y_K}^{y_{K-1}} f(t - y_k) dt \qquad (k \in \{1, \ldots, K\}),$$

this becomes

$$\det M = \int_{y_K}^{y_{K-1}} \begin{vmatrix} f(y_1 - y_1) & \cdots & f(y_1 - y_K) \\ \vdots & \ddots & \vdots \\ f(y_{K-1} - y_1) & \cdots & f(y_{K-1} - y_K) \\ f(t - y_1) & \cdots & f(t - y_K) \end{vmatrix} dt. \tag{7}$$

As f is a proper Pólya frequency function, the determinant of the translation matrix in (7) is seen to be positive for any $t \in (y_K, y_{K-1})$. Hence, $\det M > 0$. In particular, M is invertible, as claimed. It follows that $\mu_1^* = \mu_1^{**}$. Analogously, we obtain that $\mu_2^* = \mu_2^{**}$. Thus, there is at most one equilibrium in G_0. □

Proof of Lemma 2. As noted by Gröchenig (2020, p. 4), the exponential decline of $1/\xi(x + \frac{1}{2})$ on the real line ensures that its Fourier transform exists, i.e., the integral

$$\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(itx)dx}{\xi(\frac{1}{2} + x)} \qquad (-\infty < t < \infty)$$

is well-defined. Now, using Euler's formula

$$\exp(itx) = \cos(tx) + i \sin(tx),$$

as well as the functional equation $\xi(s) = \xi(1-s)$, one observes that

$$\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2} + x)} dx + \underbrace{\frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin(tx)}{\xi(\frac{1}{2} + x)} dx}_{=0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\cos(tx)}{\xi(\frac{1}{2} + x)} dx.$$

Moreover, $\xi(\frac{1}{2}) > 0$ by Lemma A.1. Thus, $f^{\#}(t) = \xi(\frac{1}{2})\tilde{f}(t)$ is indeed real-valued for $t \in \mathbb{R}$.

(i) Being the Fourier transform of an absolutely integrable function, $\tilde{f}$ is continuous on $\mathbb{R}$ (Rudin, 1974, Thm. 9.6), hence measurable. This property is inherited by $f^{\#} = \xi(\frac{1}{2})\tilde{f}$. Further, we note that $f^{\#}$, defined as a cosine integral transform, is obviously symmetric with respect to the origin.

(ii) Take some $\varepsilon > 0$. For $z \in \mathbb{C}$ such that $|\mathrm{Im}(z)| < \varepsilon$, we have

$$\left| \frac{\exp(izx)}{\xi(\frac{1}{2} + x)} \right| \leq \frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)}.$$

Moreover, from Lemma A.1(ii),

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}\left(\exp\left(|x|\left(\varepsilon - \frac{\ln x}{2}\right)\right)\right). \tag{8}$$

Focusing on the case $|x| \geq \exp(2\varepsilon)$, one observes that the left-hand side of (8) is asymptotically diminishing at an exponential rate as $|x| \to \infty$, i.e.,

$$\frac{\exp(\varepsilon |x|)}{\xi(\frac{1}{2} + x)} = \mathcal{O}(\exp(-\varepsilon |x|)). \tag{9}$$

The analytic nature of $\tilde{f}$ on the strip $|\mathrm{Im}(z)| < \varepsilon$ may now be deduced directly from (9) using the conditions put forward by Mattner (2001).[18] Again, this property is inherited by $f^{\#} = \xi(\frac{1}{2})\tilde{f}$, which proves the claim. □

The proof of Lemma 3 is prepared with some auxiliary results. The following lemma is well-known.

Lemma A.1. *The Riemann ξ-function, defined in (4), has the following properties:*

(i) $\xi > 0$ on the real line; in particular, $\xi(\frac{1}{2}) > 0$;

[16] In contrast to zero-sum games, however, this does not imply payoff equivalence across equilibria in G_0.

[17] As mentioned in the body of the paper, one can show that $y_K = 0$. However, that result is not needed in the proof.

[18] Alternatively, one may rely on Paley and Wiener (1934, Thm. I).

(ii) $\ln \xi(s) \sim \frac{1}{2} s \ln s$, *for* s *real and* $s \to \infty$;
(iii) $\mathrm{Re}\rho \in [0,1]$, *for any zero* ρ *of* ξ;[19]
(iv) $\sum_\rho \frac{1}{|\rho|^2} < \infty$, *where the sum runs over all zeros of* ξ;
(v) $\sum_\rho \frac{1}{|\rho|}$ *diverges.*

Proof. For claims (i) through (iii), see Titchmarsh (1986, pp. 29–30). For claims (iv) and (v), see Davenport (1980, Sec. 12). □

The following result, for which we could not find a reference, is used in the proof of Lemma 2.

Lemma A.2. *The "shifted"* ξ*-function admits the product representation*

$$\xi(s+\tfrac{1}{2}) = \xi(\tfrac{1}{2}) \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho - \frac{1}{2})},$$

where the product runs over the non-trivial zeros of the zeta function. Moreover, $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-2} < \infty$ *and* $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-1} = \infty$.

Proof. We start with the first claim. By Edwards (1974, Sec 2.8), the ξ-function admits the product representation

$$\xi(s) = c \cdot \prod_\rho \left(1 - \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}}\right),$$

where c is a constant, and it is understood that the factors for ρ and $1-\rho$ are paired to guarantee conditional convergence. Replacing s by $s + \frac{1}{2}$, one obtains

$$\xi(s+\tfrac{1}{2}) = c \cdot \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right).$$

Making the pairing explicit yields

$$\xi(s+\tfrac{1}{2}) = c \cdot \prod_{\mathrm{Im}\rho > 0} \left(1 - \frac{s}{\rho - \frac{1}{2}}\right)\left(1 - \frac{s}{(1-\rho) - \frac{1}{2}}\right), \tag{10}$$

where we used the fact that $\mathrm{Im}\rho$ and $\mathrm{Im}(1-\rho)$ have opposite signs. Noting that $e^{s/(\rho-\frac{1}{2})} e^{s/(1-\rho-\frac{1}{2})} = 1$, relationship (10) transforms into

$$\begin{aligned}\xi(s+\tfrac{1}{2}) &= c \cdot \prod_{\mathrm{Im}\rho > 0} \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho-\frac{1}{2})} \left(1 - \frac{s}{1-\rho - \frac{1}{2}}\right) e^{s/(1-\rho-\frac{1}{2})} \\ &= c \cdot \prod_\rho \left(1 - \frac{s}{\rho - \frac{1}{2}}\right) e^{s/(\rho-\frac{1}{2})}, \end{aligned} \tag{11}$$

where the product in (11) is, at this point, still understood to pair the factors for ρ and $1-\rho$. However, by Edwards (1974, Sec 2.5), $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-2} < \infty$. Hence, using the Weierstrass Factorization Theorem for entire functions (Conway, 1978, p. 279), the product in (11) converges locally uniformly on the complex plane. Moreover, letting $s = 0$ in (11) yields $c = \xi(\frac{1}{2})$. This completes the proof of the first claim.

The second claim has been shown above.

As for the third and final claim, we note that by Lemma A.1(iii), $\mathrm{Re}\rho \in [0,1]$. Moreover, by Lemma A.1(iv), $|\rho|^2 > \frac{1}{4}$ for all but finitely many zeros ρ.[20] Hence, for all but finitely many zeros ρ,

$$\left|\rho - \tfrac{1}{2}\right|^2 = \left|(\mathrm{Re}\rho) - \tfrac{1}{2}\right|^2 + |\mathrm{Im}\rho|^2 \le \tfrac{1}{4} + |\mathrm{Im}\rho|^2 \le 2\,|\rho|^2,$$

which implies $\left|\rho - \frac{1}{2}\right| \le \sqrt{2}\,|\rho|$. Therefore, using Lemma A.1(v), the infinite sum $\sum_\rho \left|\rho - \frac{1}{2}\right|^{-1}$ diverges. This proves the lemma. □

[19] As before, $\mathrm{Re}\rho$ and $\mathrm{Im}\rho$ denote the real and imaginary parts of the complex number ρ, respectively.

[20] Computationally, this inequality even holds for all zeros ρ, because $\mathrm{Im}\rho_1 \simeq 14.1$ for the zero ρ_1 with the smallest positive imaginary component.

Proof of Lemma 3. Suppose that RH is true. Then, by Gröchenig (2020, Thm. 3), there exists a Pólya frequency function $\Lambda(x)$ such that

$$\frac{1}{\xi(\frac{1}{2} + is)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Lambda(x) \exp(xs) dt \qquad (-\infty < s < \infty). \tag{12}$$

Applying now Mellin's inverse formula, as in Schönberg (1947, Thm. 3), one obtains

$$\Lambda(x) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{\exp(xz)dz}{\xi(\frac{1}{2} + iz)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(ixt)dt}{\xi(\frac{1}{2} + t)} \equiv \widetilde{f}(x). \tag{13}$$

Thus, replicating Gröchenig's (2020) argument, it follows that $\widetilde{f}$ is a Pólya frequency function if RH is true. We claim that $\widetilde{f}$ is even a *proper* Pólya frequency function if RH is true. We know that the "shifted" ξ is an entire function. Moreover, given our assumption regarding RH, $\rho - \frac{1}{2} = i\mathrm{Im}\rho$ is purely imaginary for each non-trivial root ρ of the zeta function. Hence, invoking Lemma A.2, we have the product representation

$$\xi(\tfrac{1}{2} + is) = \xi(\tfrac{1}{2}) \prod_\rho \left(1 - \frac{s}{\mathrm{Im}\rho}\right) e^{s/\mathrm{Im}\rho},$$

where $\sum_\rho |\mathrm{Im}\rho|^{-1} = \infty$ and $\sum_\rho |\mathrm{Im}\rho|^{-2} < \infty$. Therefore, in view of (12) and (13), Schoenberg and Whitney (1953, Thm. 1, Case 2) implies that $\widetilde{f}$ is a proper Pólya frequency function. Given that $\xi(\frac{1}{2}) > 0$ by Lemma A.1(i), the same is true for $f^{\#}$.

It remains to be shown that $f^{\#}$ is a probability density function. By evaluating the determinant of the translation matrix in the special case $n = 1$, $a_1 = t$, and $b_1 = 0$, one checks that $\widetilde{f}$ is globally positive. Therefore, from the Fourier inversion theorem (Rudin, 1974, Thm. 9.11),

$$\int_{-\infty}^{+\infty} \widetilde{f}(t) dt = \frac{1}{\xi(\frac{1}{2})}.$$

Thus, $\int_{-\infty}^{+\infty} f^{\#}(t)dt = 1$, and $f^{\#}$ is indeed a probability density function. This proves the lemma. □

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